Exercice 1. Let $x \in X$ and let $\{x_k\}_{k\in\mathbb{N}} \subset X$ be a sequence such that $x_k \underset{k\to\infty}{\longrightarrow} x$. As in Theorem 2.1.1 of the lecture notes, we can assume without loss of generality that $F_K(x) < \infty$ (lest the proof be complete). In particular, we can also assume that $x_k \in K$ for all $k \in \mathbb{N}$, and the weak sequential convergence shows that $x \in K$. Therefore, the lower semi-continuity of F applies and we deduce that

$$F_K(x) = F(x) \leq \liminf_{k\to\infty} F_K(x_k) = \liminf_{k\to\infty} F_K(x_k).$$

Finally, for all $t \in \mathbb{R}$, $X \cap \{x : F_K(x) \leq t\} = K \cap \{x : F(x) \leq t\}$ which is sequentially compact as the intersection of a sequentially compact and a sequentially closed set.

Exercice 2 (Lemma of Du Bois-Reymond)**.** We show that for all $\varphi \in C_c^\infty(\mathbb{R})$, there exists $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi' = \varphi$ if and only if $\varphi \in H$. Indeed, if $\varphi = \psi'$, using the fundamental theorem of calculus, we have

$$\int_{\mathbb{R}} \varphi(t)dt = \int_{\mathbb{R}} \psi'(t)dt = \lim_{R\to\infty} \left(\varphi(t) - \varphi(-t)\right) = 0$$

as ψ has compact support. It implies in particular that there exists $R > 0$ large enough such that for all $r \geq R$, we have

$$\int_{-r}^{r} \varphi(t)dt = \int_{-R}^{R} \varphi(t)dt = 0.$$

Therefore, we deduce that $\varphi \in H$.

Conversely, if $\varphi \in H$, define

$$\psi(t) = \int_{-\infty}^{t} \varphi(s)ds.$$

Then, $\psi \in C^\infty(\mathbb{R})$, and we claim that ψ has compact support. Indeed, as φ has compact support, there exists $R > 0$ such that $\varphi(t) = 0$ for all $|t| \geq R$. In particular, for all $t \leq -R$, we have $\psi(t) = 0$ (as we integrate an identically zero function), and for all $t \geq R$, we have

$$\psi(t) = \int_{-\infty}^{t} \varphi(s)ds = \int_{-\infty}^{R} \varphi(s)ds = \int_{\mathbb{R}} \varphi(s)ds = 0,$$

where we used that $\varphi \in H$. Therefore, the equivalence is established.

Now, if $f \in L^1_{\mathrm{loc}}(\Omega)$ satisfies the hypothesis of the exercise, it shows in particular that

$$\int_{\mathbb{R}} f(x)\varphi(x)dx = 0 \qquad \text{for all } \varphi \in H. \tag{1}$$

To complete the proof, we notice that there was a hint in the name H was given—for H is a hyperplane. In other words (1) holds for all functions belonging to a codimension 1 subspace of $C_c^\infty(\mathbb{R})$. To complete the proof, if $\theta \in C_c^\infty(\mathbb{R})$ is any function such that

$$\int_{\mathbb{R}} \theta = 1,$$

notice that for *all* $\varphi \in \mathscr{D}(\mathbb{R})$, we have

$$\varphi - \left(\int_{\mathbb{R}} \varphi\right)\theta \in H \subset \mathscr{D}(\mathbb{R}).$$

Therefore, we obtain

$$\begin{aligned}0 &= \int_{\mathbb{R}} f(x)\left(\varphi(x)-\left(\int_{\mathbb{R}}\varphi(t)dt\right)\theta(x)\right)dx = \int_{\mathbb{R}} f(x)\varphi(x)dx - \left(\int_{\mathbb{R}}\varphi(t)dt\right)\int_{\mathbb{R}} f(x)\theta(x)dx\\ &= \int_{\mathbb{R}}\left(f(x)-\int_{\mathbb{R}} f(t)\theta(t)dt\right)\varphi(x)dx.\end{aligned}$$

Therefore, using Lemma 2.2.1 from the lecture notes, we deduce that f is equal to the constant

$$C = \int_{\mathbb{R}} f(t)\theta(t) \in \mathbb{R}.$$

Exercice 3 (Approximation of Lipschitz functions)**.** Consider

$$I_n^1 = \left[-1+\frac{1}{n}, -1+\frac{2}{n}\right] \qquad I_n^2 = \left[-\frac{1}{n}+\frac{1}{n}\right] \qquad I_n^3 = \left[1-\frac{2}{n}, 1-\frac{1}{n}\right].$$

On each interval, we construct polynomial functions such that

$$\begin{aligned}
&P_n^1\left(-1\frac{1}{n}\right) = (P_n^1)'\left(-1+\frac{1}{n}\right) = 0, \qquad P_n^1\left(-1+\frac{2}{n}\right) = u\left(-1+\frac{2}{n}\right), \qquad (P_n^1)'\left(-1+\frac{2}{n}\right) = -1\\
&P_n^2\left(\pm\frac{1}{n}\right) = u\left(\pm\frac{1}{n}\right) \qquad (P_n^2)'\left(\pm\frac{1}{n}\right) = \pm 1\\
&P_n^3\left(1-\frac{1}{n}\right) = (P_n^3)'\left(1-\frac{1}{n}\right) = 0, \qquad P_n^3\left(1-\frac{2}{n}\right) = u\left(1-\frac{2}{n}\right), \qquad (P_n^3)'\left(1-\frac{2}{n}\right) = 1.
\end{aligned}$$

Then, we define

$$u_n(t) = \begin{cases} 0 & \text{for all } |t| > 1-\dfrac{1}{n}\\ P_n^i(t) & \text{for all } t \in I_n^i \;\; i=1,2,3\\ u(t) & \text{otherwise}\end{cases}$$

Since we want to interpolate those polynomial functions as at most three non-symmetric values, we can choose polynomial of degree at most 3. A computation shows that

$$\begin{cases} P_n^1(t) = \left(t+1-\dfrac{1}{n}\right)^2\left(3n^2(t+1)-8n\right)\\ P_n^2(t) = \dfrac{n}{2}\left(t^2-\dfrac{1}{n^2}\right)-1+\dfrac{1}{n}\\ P_n^3(t) = \left(t-1+\dfrac{1}{n}\right)^2\left(-3n^2(t-1)-8n\right)\end{cases}$$

furnishes the appropriate solution. The verification of the various properties is elementary and we omit it (the main step is to show that $(P_n^i)'$ is bounded, and this is easy to establish).

Exercice 4 (Euler-Lagrange equation).

1. The Euler-Lagrange equation is given by

$$\frac{d}{dt}\left(2u'(t)-4(u'(t))^3\right)=0,$$

which implies that $u'(t)-2(u'(t))^3$ is constant. However, the polynomial $X-2X^3+C$ ($C\in\mathbb{R}$) admits at most three real roots, which implies that u' must be constant. Therefore, u is an affine function, but then, the boundary conditions imply that $u=0$.

However, it is not a minimiser since $\left\{u_k : t\mapsto k(1-t^2)\right\}_{k\in\mathbb{N}}$ satisfies the boundary conditions and we easily check that

$$L(u_k)\underset{k\to\infty}{\longrightarrow}-\infty,$$

which shows that L does not admit a minimiser.

2. The Euler-Lagrange equation is given by

$$\frac{d}{dt}\left(2\,u^2(t)\,(2t-u'(t))\right)=2\,u(t)\,(2t-u'(t))^2\,.$$

A particular solution is given by

$$u(t)=\begin{cases}t^2 & \text{for all } t>0\\ 0 & \text{for all } t\leq 0.\end{cases}$$

Let us check that u is a minimiser. Indeed, $F\geq 0$, which shows that $L\geq 0$, and since $L(u)=0$, we deduce that u is a minimiser. By continuity, any C^1 minimiser must satisfy either $v(t)=0$ or $v'(t)=2t$ at every given $t\in[-1,1]$. The boundary conditions and the C^1 regularity of v imply that v takes the form

$$v(t)=\begin{cases}t^2-t_0^2 & \text{for all } t>t_0\\ 0 & \text{for all } t\leq t_0.\end{cases}$$

The boundary condition $v(1)=1$ shows that $t_0=0$, and this implies that $v=u$.